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Theoretical Methods for Estimating Moments of Inertia of trees and boles

JOHN A. STUROS

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THEORETICAL METHODS FOR ESTIMATING MOMENTS OF INERTIA OF TREES AND BOLES

John A. Sturos

The increased influx of multifunctional logging equipment into the timber industry has stimulated research in dynamic handling analysis. The designers of such equipment realize that more basic engineering data on trees and tree-length logs are necessary before dynamic conditions can be accurately analyzed. In other words, the tree must be described in engineering terms to predict the dynamic forces and moments required to mechanically handle the trees. Steinhilb and Erickson presented prediction equations for determining the weights and centers of gravity for full trees and boles.[1,2] However, no information is available on the moments of inertia for trees and boles necessary for dynamic handling analysis. This paper compares results from theoretical models for determining mass moments of inertia of trees and boles with a small sample of experimentally determined moments of inertia for red pine and aspen trees and boles.

THEORETICAL MODELS

The theoretical models presented were developed primarily in a study by Snyder and Hutula that was sponsored by the North Central Forest Experiment Station.[3] This model assumed the tree to be composed of a truncated cone of variable density (the bole) and a sphere (the crown). Comparisons between this model and the experimental data showed that better correlation was obtained with the boles than the trees. Therefore, different models of tree crowns were investigated by the author; specifically they were a right-circular cone (fig. 1) and a biconical-shaped crown (fig. 2). Both models were studied for aspen and red pine.

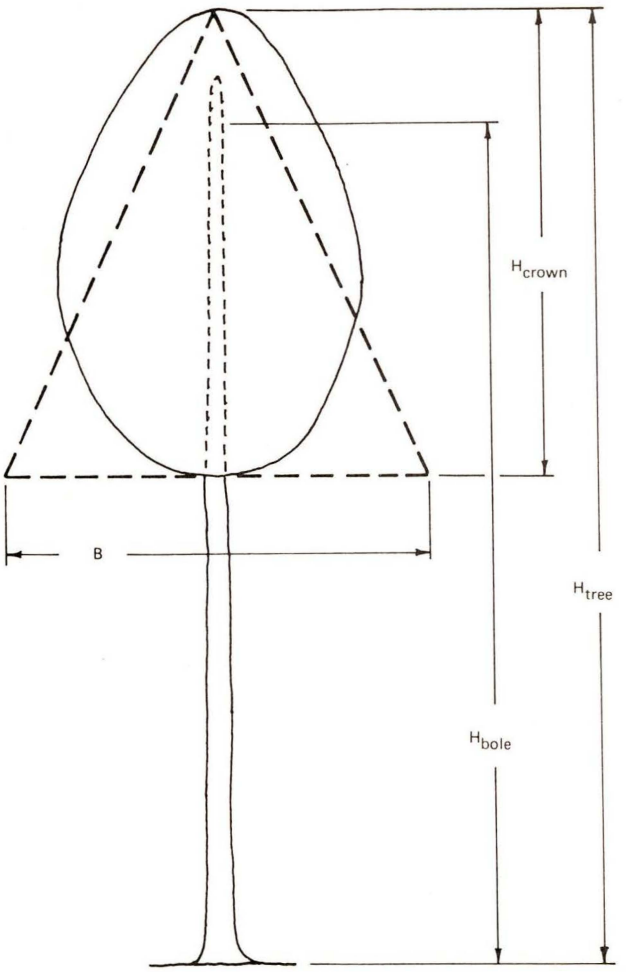

Figure 1.--Mathematical model of a tree with the bole as a truncated cone and the crown as a right circular cone.

The derivation of the theoretical models is included in Appendix B (refer also to Appendix A for the complete list of symbols). The method requires knowing the following tree parameters: d.b.h., D_{top}, H_{tree}, H_{bole}, W_{tree}, W_{bole}, C_{tree}, C_{bole}, and H_{crown}. Estimates of these variables, with the exception of H_{bole} and H_{crown}, have been

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[1] *H. M. Steinhilb and John R. Erickson. Weights and centers of gravity for quaking aspen trees and boles. USDA For. Serv. Res. Note NC-91, 4 p. North Cent. For. Exp. Stn., St. Paul, Minn. 1970.*

[2] *H. M. Steinhilb and John R. Erickson. Weights and centers of gravity for red pine, white spruce, and balsam fir. USDA For. Serv. Res. Pap. NC-75, 7 p. North Cent. For. Exp. Stn., St. Paul, Minn. 1972.*

[3] *V. Snyder and D. Hutula. Theoretical analysis of static and dynamic forces incurred in handling full trees and tree-length logs. Mich. Technol. Univ. unpubl. rep., submitted as an in-house report to Forest Eng. Lab., North Cent. For. Exp. Stn., USDA For. Serv., Houghton, Mich. 43 p.*

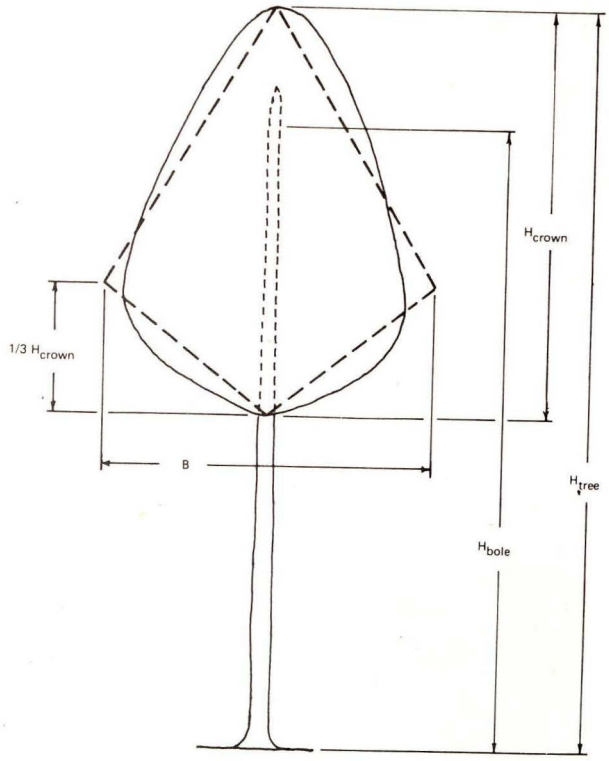

Figure 2.--Mathematical model of a tree with the bole as a truncated cone and the crown as a bicone.

published for various species including aspen and red pine.[1,2] Preliminary prediction equations for H_{bole} and H_{crown} for aspen and red pine are as follows:

Aspen:

$$H_{bole} = 1.51 + 0.8217\ (H_{tree}) + 0.0002\ (H_{tree})^2$$

$$H_{crown} = 29.64 + 0.1437\ (H_{tree}) + 0.0007\ (H_{tree})^2$$

Red pine:

$$H_{bole} = -6.08 + 0.9514\ (H_{tree}) + 0.0009\ (H_{tree})^2$$

$$H_{crown} = 25.96 - 0.0715\ (H_{tree}) + 0.0038\ (H_{tree})^2$$

EXPERIMENTAL PROCEDURE

During the summer of 1970 the sample trees were selected from adjoining pure aspen and red pine stands on a dry upland site near Alberta in Baraga County, Michigan. Data were obtained from 15 red pine and 12 aspen trees. The red pine trees ranged from 7 to 13 inches in d.b.h. and from 69 to 78 feet in height (table 1). The aspen trees ranged from 7 to 12 inches in d.b.h. and from 60 to 72 feet in height. In this paper the tree is defined as including everything above the stump. The bole is defined as the portion of the stem above the stump, delimbed and topped at a 3-inch diameter outside bark.

The mass moments of inertia for the trees and boles were determined experimentally by using the pendulum principle. The field experiments determined: (1) the weight, (2) the center of gravity, and (3) the period of oscillation of the tree or bole as it was suspended horizontally and oscillated through a small angle. Knowing the weight (W) in pounds, the distance from the center of gravity of the tree to the axis of oscillation (h) in feet, and the period of oscillation (T) in seconds, the moment of inertia about a transverse axis through the center of gravity of the tree was calculated by the following equation (refer to Appendix C):

$$\overline{I} = (T^2Wh/4\pi^2) - (Wh^2/g).$$

The d.b.h. of each sample tree was measured to the nearest 0.1 inch. The tree was then felled in the direction causing the least damage and loss of limbs, and skidded to a nearby landing. The maximum skidding distance was 300 feet, but the majority of trees were skidded only about 100 feet. The diameter outside bark of the felled tree was measured at the stump, at 2 and 4 feet from the butt, and at 8-foot intervals thereafter to the top diameter of 3 inches. The length and top diameter of any residual portion of the bole above the last 8-foot section, as well as the total overall length of the tree above the stump, were also measured. All lengths were measured to the nearest 0.1 foot.

The tree was picked up by a high-lift fork lift within 4 hours after felling. The tree weight was determined by placing a load transducer between the fork lift and a chain sling suspending the tree. The sling connections included a swivel and uniball joint to allow the tree to hang freely. A point on the tree vertically below the apex of the sling was marked as the center of gravity (fig. 3).

The period of oscillation of the tree was determined by oscillating the suspended tree through a small angle and measuring the elapsed time between the successive oscillations. The time was measured with an electric circuit and switch, the switch being a metal rod inserted into the tree. The switch opened and closed the circuit as the tree oscillated past a contact that was mounted on a vertical rod hammered into the ground (fig. 4). The closing of the switch caused a small change in voltage that was recorded on an oscillograph recorder. The recorder was used to measure time accurately between each voltage change or oscillation of the tree. The average period of oscillation used in the calculations was obtained over at least 10 oscillations. In order to decrease the amount

Table 1.--*Range of variables for aspen and red pine trees and tree-length boles*

Variable	Aspen	Red Pine
D.b.h. (in.)	7.1 to 12.1	7.0 to 13.5
Tree height (ft.)	60.50 to 72.67	69.58 to 77.75
Bole length (ft.)	45.25 to 60.58	59.00 to 69.75
Tree weight (lbs.)	555 to 1,475	582 to 2,335
Bole weight (lbs.)	485 to 1,320	520 to 1,945
Tree center of gravity (ft. from butt)	22.67 to 28.75	28.00 to 31.33
Bole center of gravity (ft. from butt)	18.25 to 23.75	23.25 to 25.42
Tree moment of inertia (lb.-ft.-sec.^{2})	3,232 to 14,726	6,569 to 24,266
Bole moment of inertia (lb.-ft.-sec.^{2})	1,558 to 8,308	4,376 to 16,102

Figure 3.--Setup showing the procedure used in weighing and locating the center of gravity of full trees

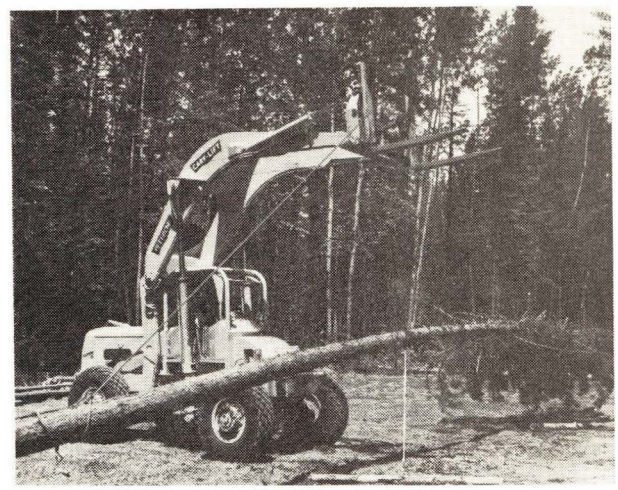

Figure 4.--Setup showing the procedure used in measuring the period of oscillation of the full tree.

of tree bending and thereby prevent the tree from contacting the ground while it oscillated, a manilla rope was tied to the butt of the tree, placed over a pulley attached to the fork lift, and then tied to the top of the tree.

After the data were obtained the tree was limbed and topped to a 3-inch diameter outside bark. The resulting bole was suspended, weighed, its center of gravity marked, and the period of oscillation measured in the same way as detailed above.

ASPEN AND RED PINE RESULTS

Equations for estimating the moments of inertia for aspen and red pine trees and boles were obtained through linear regression analyses. These equations and the associated standard errors of estimate are presented in table 2. D.b.h. was the only independent variable used. The range in height was not enough to provide information on either the value of height as an estimator or on the relationship of moments of inertia to d.b.h. over all height classes.

The 95-percent upper tolerance limits were calculated for each equation. The upper tolerance limit gives reasonable assurance (probability 0.90) that not more than 5 percent of the trees or boles from the population represented (70-foot height class) will exceed this limit. Equations and tolerance limits are shown in figures 5, 6, 7, and 8. These upper tolerance limits are offered as moment of inertia guides in the design of timber harvesting equipment.

Table 2.--*Prediction equations for the moment of inertia of red pine and aspen trees and tree-length boles*

Equation	N	Standard error of estimate	Correlation coefficient
Red pine tree:			
Y = 2,763.9 (d.b.h.) - 13,599	15	1,375	0.9728
Red pine bole:			
Y = 1,712.3 (d.b.h.) - 8,497.9		1,151	.9479
Aspen tree:			
Y = 2,514.0 (d.b.h.) - 15,239	12	1,031	.9672
Aspen bole:			
Y = 1,324.4 (d.b.h.) - 8,059.3		1,051	.8915

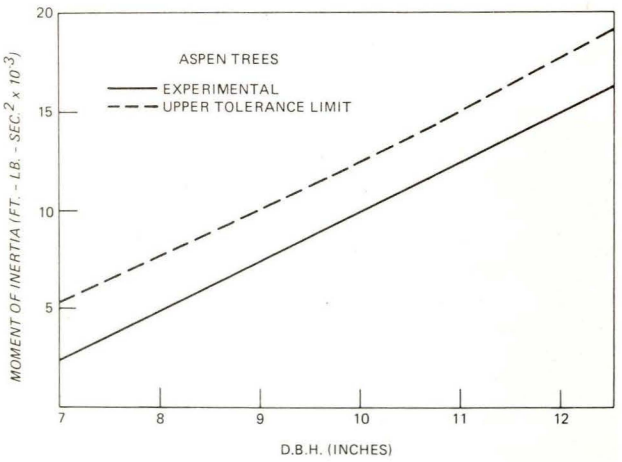

Figure 5.--Moment of inertia prediction equation for aspen trees in the 70-foot height class. The dashed line is the 95-percent upper tolerance limit.

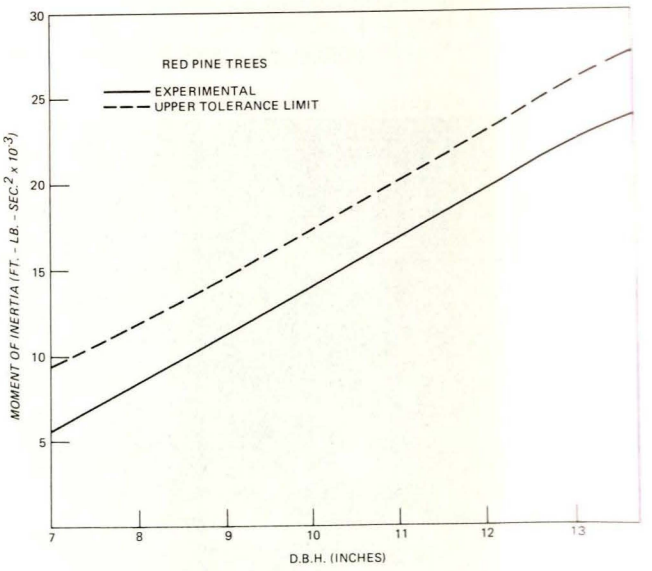

Figure 7.--Moment of inertia prediction equation plus its 95-percent upper tolerance limit for red pine trees in the 70-foot height class.

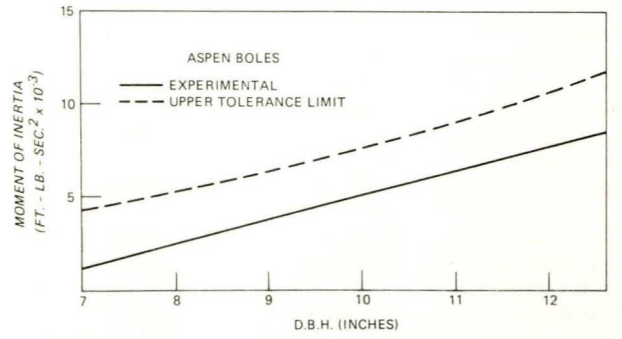

Figure 6.--Moment of inertia prediction equation plus its 95-percent upper tolerance limit for aspen boles obtained from trees in the 70-foot height class.

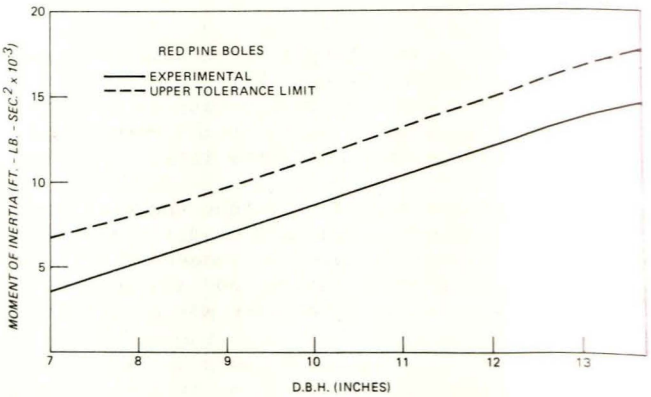

Figure 8.--Moment of inertia prediction equation plus its 95-percent upper tolerance limit for red pine boles obtained from trees in the 70-foot height class.

MODEL EVALUATION

The theoretical moment of inertia was calculated for each aspen and red pine tree and bole. Each of the tree characteristics used in the model was measured directly.

To evaluate each theoretical model the differences between the theoretical and experimental moments of inertia were analyzed statistically. A summary of these analyses is presented in table 3. In general, the theoretical tree models gave values that were not significantly different from the experimental results, but the bole theoretical models gave biased results. In all cases there was a linear relationship between the theoretical and experimental results.

Though close agreement was obtained between the theoretical and experimental results with the aspen concial tree model, the differences observed were correlated with d.b.h. With the aspen biconical tree model the theoretical moments were significantly higher than the experimental moments but approximated the upper tolerance limits of the experimental data. The bias was consistent throughout the data; the differences were not related to tree characteristics.

The conical red pine tree model gave moments of inertia that were not signficantly different from the experimental; there were large differences from tree to tree (range in the differences was from -3,238 to +5,015), yielding a large standard error. The differences between theoretical and experimental results were correlated with tree height.

The theoretical model consistently overestimated the moment of inertia of the aspen and red pine boles but never exceeded the upper tolerance limits of the experimental data. These differences were relatively constant throughout the range of the moments of inertia. For the aspen boles they were correlated with bole length. For the red pine boles the differences were correlated with both d.b.h. and bole length.

These analyses indicate the limitations when using the theoretical models for estimating the moments of inertia for trees in the 70-foot height class. A demonstration of the utility of the model was conducted on a taller tree. The experimental and theoretical moments of inertia for an 88-foot aspen tree with 18.8-inch d.b.h. were calculated and close agreement found as indicated in the tabulation:

	Experimental value ($ft.-lb.-sec^2$)	*Conical-shaped model* ($ft.-lb.-sec^2$)	*Biconical-shaped model* ($ft.-lb.-sec^2$)
Tree	53,783	55,261	63,087
Bole	30,581	34,485	34,485

UTILIZATION FOR OTHER SPECIES

As mentioned in the introduction, basic engineering data on the moments of inertia of trees and boles are needed to enable a dynamic handling analysis for a specific machine design. Equations for estimating the moments of inertia

Table 3.--*Summary of the statistical analyses made in comparing the theoretical and experimental values of moment of inertia*

	Aspen			Red pine	
	Trees			Trees	
	Conical-shaped crown	Biconical-shaped crown	Boles	Conical-shaped crown	Boles
Number of samples	10	10	12	12	13
Mean difference (theoretical - experimental)	-80	1,191	1,147	744	1,314
Standard deviation of the differences	1,030	975	349	2,274	836
Standard error of the mean difference	326	308	101	657	232
Confidence interval about the mean with one standard error	-406 245	883 1,499	1,046 1,248	88 1,401	1,082 1,546
$t_{calculated}$	-0.25	3.86	11.40	1.13	5.67
Correlation coefficient: differences vs. d.b.h.	-.56	.08	.29	.31	.49
Correlation coefficient: differences vs. tree height or bole length	-.25	.23	.52	.66	.73
Linear regression: experimental vs. theoretical					
Slope	1.05	.93	.94	.81	.86
Intercept	404	400	779	2,462	156
Correlation coefficient	.96	.97	.99	.94	.98

for aspen and red pine trees and boles, presented earlier, can be used for trees of those species, within the range of data shown in table 1. For trees outside these ranges and for trees of other species, the theoretical models could be used to obtain estimates. The theoretical approach is necessary because no other information is available. The models should work best with species that are similar to aspen or red pine.

A dynamic analysis method in which the above data can be used has been developed by Sturos and Mattson.[4] This technique utilizes a computer program developed by IBM Corporation called Continuous System Modeling Program (CSMP). Equations relating the tree weight, center of gravity, and moment of inertia to d.b.h. can all be incorporated into the program. The dynamic forces and moments required in handling any specified size of tree through any specified handling mode can be determined.

SUMMARY

Estimating equations for the moments of inertia of trees and boles and their upper tolerance limits have been presented for a limited aspen and red pine tree and bole population. The test of the theoretical models on the aspen and red pine data shows they have some promise but more research is necessary. The theoretical models gave moments of inertia for the aspen and red pine trees that were not significantly different statistically at the 0.05 level from the actual moments of inertia. Overestimation of the moments of inertia of the boles was evident, but the amount was consistent through the range of d.b.h. tested. Correlation of the differences between the experimental and theoretical results to tree characteristics is evident. Some improvements were obtained in the tree model by using the conical crown form rather than the biconical form.

The theoretical methods presented are only a start in obtaining more realistic design data and are presented here for consideration by others. Though certain simplifying assumptions were made, the models give conservative figures on which the designer can specify his first prototype.

[4] *John A. Sturos and James A. Mattson. Dynamic analysis of handling full trees and tree-length logs. Am. Soc. Agric. Eng. Trans. (In press.)*

APPENDIX A

Legend of Symbols

A = Cross-sectional area of the bole (ft.^{2}).

C_{tree} = Distance from butt to center of gravity of full tree (ft.).

C_{bole} = Distance from butt to center of gravity of tree-length bole (ft.).

C_{crown} = Distance from butt to center of gravity of crown (ft.).

d = Bole diameter at arbitrary point (ft.).

d.b.h. = Bole diameter at breast height (in.).

D_{top} = Bole diameter at top (in.).

g = Gravitational acceleration, 32.2 ft./sec.^{2}.

h = Distance from center of gravity of tree or bole to axis of oscillation (ft.).

H_{tree} = Height of full tree with crown (ft.).

H_{bole} = Height of tree-length bole (ft.).

H_{crown} = Height of crown (ft.).

IX_{bole} = Moment of inertia of bole about transverse axis through center of gravity of bole (ft.-lb.-sec.^{2}).

IZ_{bole} = Moment of inertia of bole about longitudinal axis (ft.-lb.-sec.^{2}).

IX_{tree} = Moment of inertia of full tree about transverse axis through center of gravity of full tree (ft.-lb.-sec.^{2}).

IZ_{tree} = Moment of inertia of full tree about longitudinal axis (ft.-lb.-sec.^{2}).

I_{crown} = Moment of inertia of crown about transverse axis throuch center of gravity of crown (ft.-lb.-sec.^{2}).

ρ = Density of bole (lb./ft.^{3}).

T = Period of oscillation of the full tree or tree-length bole (sec.).

W_{tree} = Weight of full tree (lbs.).

W_{bole} = Weight of tree-length bole (lbs.).

W_{crown} = Weight of crown (lbs.).

B = Width of theoretical tree crown.

APPENDIX B

Theoretical Method

If the bole of the tree is assumed to be a truncated cone, the diameter d varies linearly along the bole length and is given by:

$$d = C_1 - C_2 Z$$

where Z is measured from the butt end. The coefficients C_1 and C_2 are determined by using the following conditions:

$d = d.b.h./12$ at $Z = 5$ (breast height)
$d = D_{top}/12$ at $Z = H_{bole}$ (top of bole).

The factors 1/12 in the equations are used because d.b.h. and D_{top} are in inches and d is in feet. Using the above equations, the coefficients C_1 and C_2 are obtained as:

$$C_1 = \frac{(H_{bole} \times d.b.h.) - 5 D_{top}}{12 (H_{bole} - 5)}$$

$$C_2 = \frac{d.b.h. - D_{top}}{12(H_{bole}-5)} .$$

The density ρ of the bole is taken to vary linearly along the bole length in the form:

$$\rho = K_1 - K_2 Z .$$

The coefficients K_1 and K_2 are determined by using the known bole weight and CG location.

$$W_{bole} = \int_0^{H_{bole}} \rho A dz = \frac{\pi}{4} \int_0^{H_{bole}} (K_1 - K_2 Z)(C_1 - C_2 Z)^2 dz$$

$$= \frac{\pi H_{bole}}{4} \left[C_1^2 - C_1 C_2 H_{bole} + \frac{1}{3} C_2^2 (H_{bole})^2\right] K_1$$

$$- \frac{\pi (H_{bole})^2}{4} \left[\frac{1}{2} C_1^2 - \frac{2}{3} C_1 C_2 H_{bole} + \frac{1}{4} C_2^2 (H_{bole})^2\right] K_2$$

$$C_{stem} =$$

$$\frac{1}{W_{bole}} \int_0^{H_{bole}} \rho A z dz = \frac{\pi}{4 W_{bole}} \int_0^{H_{bole}} (K_1 - K_2 Z)(C_1 - C_2 Z)^3$$

$$= \frac{\pi (H_{bole})^2}{4 W_{bole}} \left[\frac{1}{2} C_1^2 - \frac{2}{3} C_1 C_2 H_{bole} + \frac{1}{4} C_2^2 (H_{bole})^2\right] K_1$$

$$- \frac{\pi (H_{bole})^3}{4 W_{bole}} \left[\frac{1}{3} C_1^2 - \frac{1}{2} C_1 C_2 H_{bole} + \frac{1}{5} C_2^2 (H_{bole})^2\right] K_2 .$$

The above two equations are two linear-algebraic equations which can be solved for K_1 and K_2.

The mass moment of inertia of the bole about a transverse axis can be computed by treating the bole as a long slender rod with a variable density and cross section. Using the integral definition of moment of inertia and the transfer-axis theorem, the moment of inertia of the bole about transverse axis x through the bole CG is determined to be:

$$IX_{bole} =$$

$$\frac{1}{g} \int_0^{H_{bole}} \rho A Z^2 dz - \frac{W_{bole} (C_{bole})^2}{g}$$

$$= \frac{\pi}{4g} \int_0^{H_{bole}} (K_1 - K_2 z)(C_1 - C_2 z)^2 z^2 dz - \frac{W_{bole} (C_{bole})^2}{g}$$

$$= \frac{\pi K_1 (H_{bole})^3}{4g} \left[\frac{1}{3} C_1^2 - \frac{1}{2} C_1 C_2 H_{bole} + \frac{1}{5} C_2^2 (H_{bole})^2\right]$$

$$- \frac{\pi K_2 (H_{bole})^4}{4g} \left[\frac{1}{4} C_1^2 - \frac{2}{5} C_1 C_2 H_{bole} + \frac{1}{6} C_2^2 (H_{bole})^2\right]$$

$$- \frac{W_{bole} (C_{bole})^2}{g} .$$

Two types of crowns were investigated, namely a right circular cone and a biconical shape as shown in figures 1 and 2. The moment of inertia of a right circular cone about its own center of gravity is:

$$I_{crown} = \frac{W_{crown}}{g} \left(\frac{3}{80} B^2 + \frac{3}{80} H^2_{crown}\right) .$$

Assume: $B = H_{crown}$,

then: $$I_{crown} = \frac{3}{40} \frac{W_{crown}}{g} (H^2_{crown})$$

$$C_{crown} = H_{tree} - 3/4\ H_{crown} .$$

The biconical-shaped crown studied was made up of two unequal right-circular cones with a common base, the top cone being twice as high as the bottom one as shown in figure 2. The center of gravity of this bicone with respect to the base of the tree is:

$$C_{crown} = H_{tree} - 7/12\ H_{crown} .$$

The moment of inertia of this bicone about its own center of gravity is:

$$I_{crown} = \frac{W_{crown}}{g} \left(\frac{3}{80} B^2 + \frac{19}{720} H^2_{crown}\right) .$$

Assume: $B = H_{crown}$,

then: $$I_{crown} = \frac{23}{360} \frac{W_{crown}}{g} H^2_{crown} .$$

The moment of inertia of a full tree about a transverse axis through the CG of the full tree is given by:

$$IX_{tree} = IX_{bole} + \frac{W_{bole}}{g} (C_{tree} - C_{bole})^2 + I_{crown} + \frac{W_{crown}}{g} (C_{tree} - C_{crown})^2 .$$

APPENDIX C

Experimental Method

The moment of inertia of a compound pendulum (I') about its axis of oscillation is given by the following equation:[5]

$$I' = T^2Wh/4\pi^2$$

where: T = period of oscillation (seconds),
W = weight of pendulum (pounds), and
h = distance from the center of gravity of the pendulum to the axis of oscillation (feet).

Therefore, by using the parallel-axis theorem, the moment of inertia of a pendulum (tree or bole) about its center of gravity ($\bar{I}$) is given by the following equation:

$$\bar{I} = I' - Wh^2/g = (T^2Wh/4\pi^2) - (Wh^2/g)$$

where: g = acceleration due to gravity (feet/second^2).

The effect of the amplitude of the oscillations and the damping at the point of suspension are assumed negligible.

[5] *E. Hausman and E. P. Slack. Physics. Ed. 2, p. 159. New York: D. Van Nostrand Co. 1939.*

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Sturos, John A.
1973. Theoretical methods for estimating moments of inertia of trees and boles. North Cent. For. Exp. Stn., St. Paul, Minn. 10 p., illus. (USDA For. Serv. Res. Pap. NC-96)

Presents a theoretical method for estimating the mass moments of inertia of full trees and boles about a transverse axis. Estimates from the theoretical model compared closely with experimental data on aspen and red pine trees obtained in the field by the pendulum method. The theoretical method presented may be used to *estimate* the mass moments of inertia of other tree species for which field data are not available.

OXFORD: 307:377:U531.23. KEY WORDS: dynamic handling, equipment design, harvesting, engineering.

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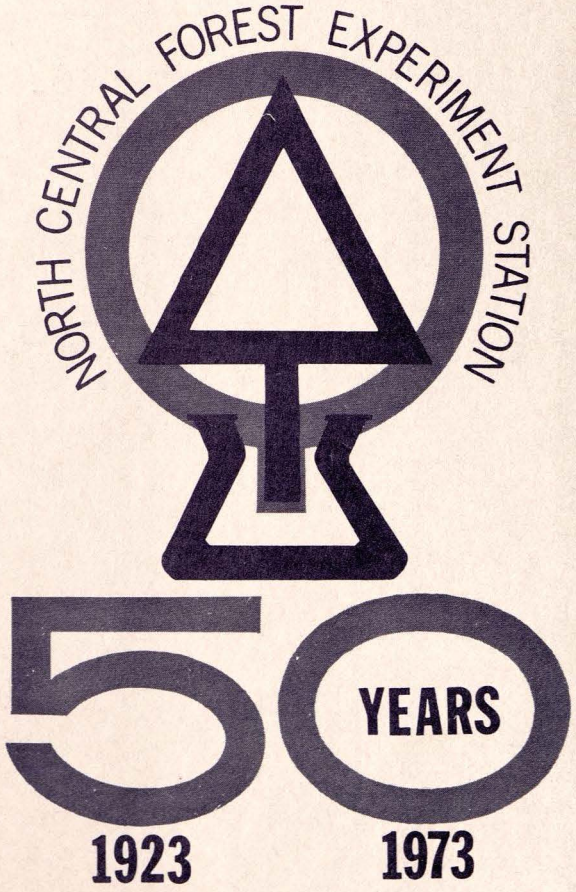
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